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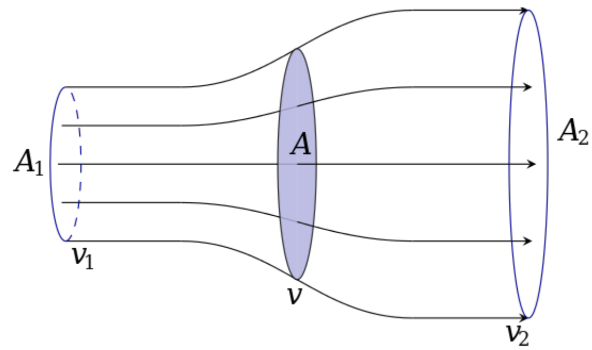
Betz's law

Betz's law indicates the maximum power that can be extracted from the wind, independent of the design of a wind turbine in open flow. It was published in 1919, by the German physicist Albert Betz.[1] The law is derived from the principles of conservation of mass and momentum of the air stream flowing through an idealized "actuator disk" that extracts energy from the wind stream. According to Betz's law, no turbine can capture more than 16/27 (59.3%) of the kinetic energy in wind. The factor 16/27 (0.593) is known as Betz's coefficient. Practical utility-scale wind turbines achieve at peak 75% to 80% of the Betz limit.[2][3]



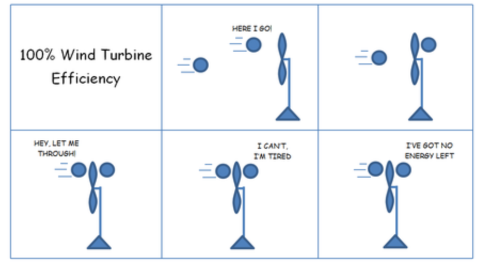
Schematic of fluid flow through a disk-shaped actuator. For a constant density fluid, cross sectional area varies inversely with speed.

The Betz limit is based on an open disk actuator. If a diffuser is used to collect additional wind flow and direct it through the turbine, more energy can be extracted, but the limit still applies to the cross-section of the entire structure.

Contents

Concepts

Betz's law applies to all Newtonian fluids, including wind. If all of the energy coming from wind movement through a turbine were extracted as useful energy, the wind speed afterwards would drop to zero. If the wind stopped moving at the exit of the turbine, then no more fresh wind could get in; it would be blocked. In order to keep the wind moving through the turbine, there has to be some wind movement, however small, on the other side with some wind speed greater than zero. Betz's law shows that as air flows through a certain area, and as wind speed slows from losing energy to extraction from a turbine, the airflow must distribute to a wider area. As a result, geometry limits any turbine efficiency to a maximum of 59.3%.



Simple cartoon of two air molecules shows why wind turbines cannot actually run at 100% efficiency

Independent discoveries

British scientist Frederick W. Lanchester derived the same maximum in 1915. The leader of the Russian aerodynamic school, Nikolay Zhukowsky, also published the same result for an ideal wind turbine in 1920, the same year as Betz did.[4] It is thus an example of Stigler's Law which posits that no scientific discovery is named after its actual discoverer.

Economic relevance

The Betz limit places an upper bound on the annual energy that can be extracted at a site. Even if a hypothetical wind blew consistently for a full year, no more than the Betz limit of the energy contained in that year's wind could be extracted. Since wind speed varies, the annual capacity factor of a wind site is around 25% to 60% of the energy that would be generated with constant wind. The 2016 average for the United States was 34.7%

Essentially increasing system economic efficiency results from increased production per unit, measured per square meter of vane exposure. An increase in system efficiency is required to bring down the cost of electrical power production measured in kW. Efficiency increases may be the result of engineering of the wind capture devices, such as the configuration and dynamics of wind turbines, that may push the power generation from these systems into higher levels of the Betz limit. System efficiency increases in power application, transmission or storage may also contribute to a lower cost of power per unit.

Some designs have claimed to approach the Betz constant and even to surpass it, but none have been proven to do so.[5][6]

Proof

The Betz Limit is the maximum possible energy that may be derived by means of an infinitely thin rotor from a fluid flowing at a certain speed.[7]

In order to calculate the maximum theoretical efficiency of a thin rotor (of, for example, a windmill) one imagines it to be replaced by a disc that withdraws energy from the fluid passing through it. At a certain distance behind this disc the fluid that has passed through flows with a reduced velocity.[7]

Assumptions

1. The rotor does not possess a hub and is ideal, with an infinite number of blades which have no drag. Any resulting drag would only lower this idealized value.
2. The flow into and out of the rotor is axial. This is a control volume analysis, and to construct a solution the control volume must contain all flow going in and out, failure to account for that flow would violate the conservation equations.
3. The flow is non-compressible. Density remains constant, and there is no heat transfer.
4. Uniform thrust is exerted on the disc or rotor.

Application of conservation of mass (continuity equation)

Applying conservation of mass to this control volume, the mass flow rate (the mass of fluid flowing per unit time) is given by:

$$\dot{m} = \rho A_1 v_1 = \rho S v = \rho A_2 v_2$$

where v_1 is the speed in the front of the rotor and v_2 is the speed downstream of the rotor, and v is the speed at the fluid power device. ρ is the fluid density, and the area of the turbine is given by S and A_1 and A_2 are the area of the fluid before and after reaching the turbine.

So the density times the area and speed should be equal in each of the three regions, before, while going through the turbine and afterwards.

The force exerted on the wind by the rotor is the mass of air multiplied by its acceleration. In terms of the density, surface area and velocities, this can be written:

$$\begin{aligned} F &= ma \\ &= m\frac{dv}{dt} \\ &= \dot{m}\,\Delta v \\ &= \rho S v (v_1 - v_2) \end{aligned}$$

Power and work

The work done by the force may be written incrementally as

$$dE = F \cdot dx$$

and the power (rate of work done) of the wind is

$$P = \frac{dE}{dt} = F \cdot \frac{dx}{dt} = F \cdot v$$

Now substituting the force F computed above into the power equation will yield the power extracted from the wind:

$$P = \rho \cdot S \cdot v^2 \cdot (v_1 - v_2)$$

However, power can be computed another way, by using the kinetic energy. Applying the conservation of energy equation to the control volume yields

$$\begin{aligned} P &= \frac{\Delta E}{\Delta t} \\ &= \frac{1}{2} \cdot \dot{m} \cdot (v_1^2 - v_2^2) \end{aligned}$$

Looking back at the continuity equation, a substitution for the mass flow rate yields the following

$$P = \frac{1}{2} \cdot \rho \cdot S \cdot v \cdot (v_1^2 - v_2^2)$$

Both of these expressions for power are completely valid, one was derived by examining the incremental work done and the other by the conservation of energy. Equating these two expressions yields

$$P = \frac{1}{2} \cdot \rho \cdot S \cdot v \cdot (v_1^2 - v_2^2) = \rho \cdot S \cdot v^2 \cdot (v_1 - v_2)$$

Examining the two equated expressions yields an interesting result, namely

$$\frac{1}{2} \cdot (v_1^2 - v_2^2) = \frac{1}{2} \cdot (v_1 - v_2) \cdot (v_1 + v_2) = v \cdot (v_1 - v_2)$$

or

$$v = \frac{1}{2} \cdot (v_1 + v_2)$$

Therefore, the wind velocity at the rotor may be taken as the average of the upstream and downstream velocities. (This is arguably the most counter-intuitive stage of the derivation of Betz's law.)

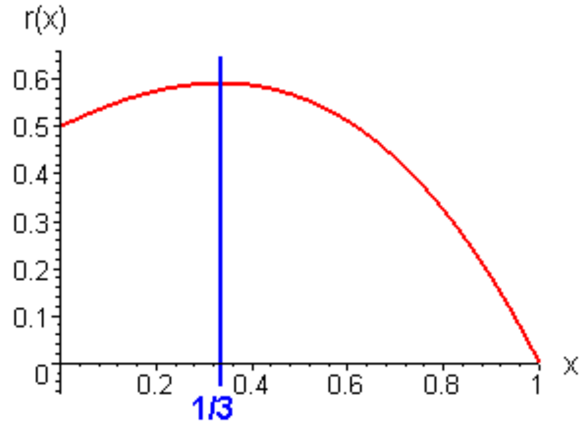
Betz's law and coefficient of performance

Returning to the previous expression for power based on kinetic energy:

$$\begin{aligned} \dot{E} &= \frac{1}{2} \cdot \dot{m} \cdot \left(v_1^2 - v_2^2\right) \\ &= \frac{1}{2} \cdot \rho \cdot S \cdot v \cdot \left(v_1^2 - v_2^2\right) \\ &= \frac{1}{4} \cdot \rho \cdot S \cdot (v_1 + v_2) \cdot \left(v_1^2 - v_2^2\right) \end{aligned}$$

$$= \frac{1}{4} \cdot \rho \cdot S \cdot v_1^3 \cdot \left(1 - \left(\frac{v_2}{v_1}\right)^2 + \left(\frac{v_2}{v_1}\right) - \left(\frac{v_2}{v_1}\right)^3\right).$$

By differentiating $\dot{E}$ with respect to $\frac{v_2}{v_1}$ for a given fluid speed v_1 and a given area S one finds the *maximum* or *minimum* value for $\dot{E}$. The result is that $\dot{E}$ reaches maximum value when $\frac{v_2}{v_1} = \frac{1}{3}$.



The horizontal axis reflects the ratio v_2/v_1, the vertical axis is the "power coefficient [1] (https://web.archive.org/web/20091031151206/http://www.talentfactory.dk/en/tour/wres/cp.htm)" C_p

Substituting this value results in:

$$P_{\text{max}} = \frac{16}{27} \cdot \frac{1}{2} \cdot \rho \cdot S \cdot v_1^3.$$

The power obtainable from a cylinder of fluid with cross sectional area S and velocity v_1 is:

$$P = C_{\text{p}} \cdot \frac{1}{2} \cdot \rho \cdot S \cdot v_1^3.$$

The reference power for the Betz efficiency calculation is the power in a moving fluid in a cylinder with cross sectional area S and velocity v_1:

$$P_{\text{wind}} = \frac{1}{2} \cdot \rho \cdot S \cdot v_1^3.$$

The *"power coefficient*[8]*"* C_{p} (= P/P_{wind}) has a maximum value of: $C_{\text{p.max}}$ = 16/27 = 0.593 (or 59.3%; however, coefficients of performance are usually expressed as a decimal, not a percentage).

Modern large wind turbines achieve peak values for C_{p} in the range of 0.45 to 0.50,[2] about 75% to 85% of the theoretically possible maximum. In high wind speed where the turbine is operating at its rated power the turbine rotates (pitches) its blades to lower C_{p} to protect itself from damage. The power in the wind increases by a factor of 8 from 12.5 to 25 m/s, so C_p must fall accordingly, getting as low as 0.06 for winds of 25 m/s.

Understanding the Betz results

Intuitively, the speed ratio of [V_2/V_1 = 0.333] between outgoing and incoming wind, leaving at about a third of the speed it came in, would imply higher losses of kinetic energy. But since a larger area is needed for slower moving air, energy is conserved.

All energy entering the system is taken into consideration, and local "radial" kinetic energy can have no effect on the outcome, which is the final energy state of the air leaving the system, at a slower speed, larger area and accordingly its lower energy can be calculated.

The last step in calculating the Betz efficiency C_p is to divide the calculated power extracted from the flow by a reference power value. The Betz analysis uses for its power reference, reasonably, the power of air upstream moving at V_1 contained in a cylinder with the cross sectional area of the rotor (S).

Points of interest

The Betz limit has no dependence on the geometry of the wind extraction system, therefore S may take any form provided that the flow travels from the entrance to the control volume to the exit, and the control volume has uniform entry and exit velocities. Any extraneous effects can only decrease the performance of the system (usually a turbine) since this analysis was idealized to disregard friction. Any non-ideal effects would detract from the energy available in the incoming fluid, lowering the overall efficiency.

Some manufacturers and inventors have made claims of exceeding the limit by using nozzles and other wind diversion devices, usually by misrepresenting the Betz limit and calculating only the rotor area and not the total input of air contributing to the wind energy extracted from the system.

Modern development

In 1934 H. Glauert derived the expression for turbine efficiency, when the angular component of velocity is taken into account, by applying an energy balance across the rotor plane.[9] Due to the Glauert model, efficiency is below the Betz limit, and asymptotically approaches this limit when the tip speed ratio goes to infinity.

In 2001, Gorban, Gorlov and Silantyev introduced an exactly solvable model (GGS), that considers non-uniform pressure distribution and curvilinear flow across the turbine plane (issues not included in the Betz approach).[10] They utilized and modified the Kirchhoff model,[11] which describes the turbulent wake behind the actuator as the `degenerated' flow and uses the Euler equation outside the degenerate area. The GGS model predicts that peak efficiency is achieved when the flow through the turbine is approximately 61% of the total flow which is very similar to the Betz result of 2/3 for a flow resulting in peak efficiency, but the GGS predicted that the peak efficiency itself is much smaller: 30.1%.

Recently, viscous computations based on computational fluid dynamics (CFD) were applied to wind turbine modelling and demonstrated satisfactory agreement with experiment.[12] Computed optimal efficiency is, typically, between the Betz limit and the GGS solution.

For VAWT, some recent theoretical research has shown that active lift turbines can deliver power coefficients greater than defined by Betz for classical vertical axis Darrieus type turbines.[13] For a standard Darrieus turbine, the normal force creates compressive stress and extension of the arms. The active lift turbine is partially transferring these normal forces thanks to the crank rod system into kinetic energy.[14]

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External links

- The Betz limit - and the maximum efficiency for horizontal axis wind turbines (http://www.wind-power-program.com/betz.htm)
- Pierre Lecanu, Joel Breard, Dominique Mouazé. Betz limit applied to vertical axis wind turbine theory (https://hal.inria.fr/hal-01300531/)